Ambiguous Networks

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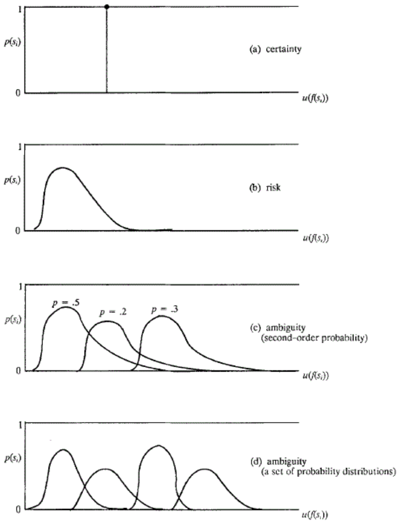
From **Gabrieli 2011**,

> *"[...] the increase in the percentage of 2-nodes SCCs especially after August 2007 might represent a sort of reversion to less efficient times; in this case, however, the banks were so wary of potential counterparts that they do not trust selected small groups of banks but prefer to be linked to just one other bank at a time[...]."*

Outline

- We show that under specific conditions, the complexity of a network structure describing relationships between agents is a source of "*uncertainty/ambiguity*".
- The extent of the *ambiguity* is related to the network architecture.
- The agents are assumed able to strategically rewire their connections.

What is *ambiguity*?



Overview

Multiple priors as "natural" response to perceived ambiguity (Savage, 1972)

- **Maxmin Expected Utility (MEU)**, of Gilboa and Schmeidler (1989)
- the **"Smooth" Ambiguity Model (KMM)**, of Klibanoff, Marinacci and Mukerji (2005)
- the **Variational Preferences Model**, of Maccheroni, Marinacci and Rustichini (2006)
  - the **Mean-Dispersion Preference Model** (Grant and Polak, 2008)
  - the **Confidence Function Model** (Chateauneuf and Faro, 2008)
  - the **Uncertainty Averse Preference Model** (Cerreia-Vioglio et al, 2008)
- the **Choquet Expected Utility Model (CEU)**, of Schmeidler (1989) and the related to the non-additive probability or capacity
- **Vector Expected Utility (VEU)**, of Siniscalchi (2009)

Problem description

- A DM belonging to a network of $N > 2$ connected players faces a stochastic outcome *directly* related to the outcomes of his direct peers.
- The indirect ties with the rest of the players create *ambiguity* over the "right" probability distribution to consider.
- The DM can rewire strategically his links in order to maximize his expected utility, given the choices of the other players and a positive marginal cost per-link.

The model

- Network $G = (N, L)$ of N nodes and L links.
- The contingency state of each direct node partner j is $\theta_j = \{H, L\} \in \Theta_J$, where the player $j \in \mathcal{N}_i^1$, with $\mathcal{N}_i^1 = \{j \in N : l_{ij} \in L\}$ is the subset of nodes distant 1 link from i.
- If $\mathcal{N}_i^{(2)} = \emptyset$, where $\mathcal{N}_i^{(2)}$ defines the set of nodes distant 2 links from i, the node i will assign a unique exogenous probability function $\rho \in [0, 1]$ to each event belonging to θ_j such that $Pr(\theta_j) = 1$.
- If $\exists j : \mathcal{N}_i^{(2)} \neq \emptyset$, the node i will use the composition of the subset of nodes connected to this j to form prior beliefs over θ_j. In particular, the subset of nodes neighbors of j adds a new list of contingencies $\theta_{jj'}$ for all the j'(s) composing the subset of direct neighbors of j, determining which event from θ_j the node i is expecting to observe.

The model

- *Implication mapping* $\Gamma_j(w_n)$, where $\Gamma_j(w_n) \in 2^{\theta_j} \setminus \emptyset$, and w_n defines the nth specific event belonging to $\theta_{jj'}$.
- i knows the probabilities associated to each w_n, $Pr(w_n)$, described by a subjective prior μ over Δ, the set of possible probabilities π over the state space Θ_J, and such that $\sum_{n=1}^{N} Pr(w_n) = 1$.
- The node i is ascribing beliefs $P^*(\cdot)$, where for any $X \in 2^{\theta_j} \setminus \emptyset$, $P^*(X) = \{\sum_{n=1}^{N} Pr(w_n) \mid \Gamma_j(w_n) \subseteq X\}$, or also that, if $\Gamma_j(w_n)$ is not singleton, then $P^*(X)$ will be consequently non-additive.

The model

- Following **Klibanoff et al. (2005)**, we assume preferences represented by a functional of the double expectational form, i.e. the ambiguity is modeled considering second-order probabilities,

$$V(f_g) = \int_{\Delta} \phi\left(\int_{\Theta_J} u(f_g) d\pi\right) d\mu \equiv \mathbb{E}_{\mu}\phi(\mathbb{E}_{\pi} u \circ g) \quad (1)$$

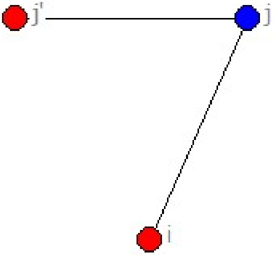
  - $\Theta_J \ni \theta_k$ set of contingencies or states
  - π is a probability distribution over Θ_J
  - f_g is a "network-act" yielding state contingent payoffs $f_g(s)$
  - u is a von Neumann-Morgenstern utility function
  - ϕ is a map from reals to reals, or alternatively is a second-order utility function
  - μ a *subjective probability* or second-order probability over Δ

Network acts

f_g is defined as a "*network act*", since it describes a structural-choice that together with μ induces a specific distribution $\mu_{\int}$.
Formally, a network-act is a Σ-measurable map $f_g : \Theta_J \to \Delta(X)$, where $\Delta(X)$ is the set of all the probability distributions (lotteries) on the outcomes X with finite support, closed under mixtures, i.e. convex combinations.

Example

Consider the Line-network g_{i2} composed by 3 nodes,



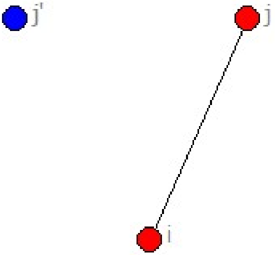
The player i will expect the following potential events combination

$$\{H_j H_{j'}, H_j L_{j'}, L_j H_{j'}, L_j L_{j'}\}$$

where each element of the set is defined as w_k and ascending ordered. Assume the following *"implication mapping"*

$$w_1 \rightarrow \{H_j\} \quad w_2 \rightarrow \{H_j, L_j\} \quad w_3 \rightarrow \{H_j, L_j\} \quad w_4 \rightarrow \{L_j\}$$

Consider the network g_{i1} composed by two components, a *dyad* of 2 nodes and an isolated node, j'.



the player i considers the node j as not-influenced by j', i.e. i faces a unique probability distributions over the state space Θ_J . We can represent the resulting environment in the following table.
The measure μ assigns equal probability $1/4$ to each π_k with k from 1 to 4,

	$H_j H_{j'} \times \frac{1}{4}$	$H_j L_{j'} \times \frac{1}{4}$	$L_j H_{j'} \times \frac{1}{4}$	$L_j L_{j'} \times \frac{1}{4}$
	2	2,1	2,1	1
	2	2	1	1

We assume a function u CRRA (normalized) and given by

$$u(x) = \begin{cases} 1 + \frac{x^{1-\rho}-1}{2^{1-\rho}-1}, & \rho \geq 0, \rho \neq 1 \\ 1 + \frac{ln(x)}{ln(2)}, & \rho = 1 \end{cases}$$

The function ϕ displays constant ambiguity aversion, where a is the coefficient of ambiguity aversion,

$$\phi(x) = \begin{cases} \frac{1-e^{-ax}}{1-e^{-a}}, & a > 0 \\ x, & a = 0 \end{cases}$$

Definition

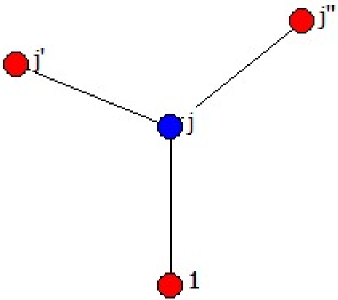
A network structure g is UNAMBIGUOUS for a node $i \in g$ if and only if $\forall w \in \Omega$, $\Gamma(w)$ the implication mapping is a singleton set. Otherwise g is AMBIGUOUS.

Lemma

Under the assumptions above, if an ambiguity averse DM is using a decision rule, or a $k \in [\frac{n}{2}+1, n]$, such that $\triangle K_i > 0$, increasing the number of peers belonging to $\mathcal{N}_{ij}^{t}$ with $t \geq 2$, he will face higher ambiguity.

Example

Consider now the following 4-nodes *Star-shaped* network,



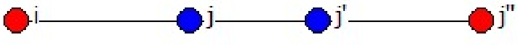
The set of all the contingencies that i associates to the outcomes of the j's peers is the following

$$\{H_j H_{j'} H_{j''}, H_j H_{j'} L_{j''}, H_j L_{j'} H_{j''}, H_j L_{j'} L_{j''},$$
$$L_j H_{j'} H_{j''}, L_j H_{j'} L_{j''}, L_j L_{j'} H_{j''}, L_j L_{j'} L_{j''}\}$$

with the following implication mapping,

$w_1 \rightarrow \{H_j\} \quad w_8 \rightarrow \{L_j\} \quad w_{k\in[2,7]} \rightarrow \{H_j, L_j\}$

Let's consider now the following 4-nodes *Line-shaped* network,



The player i faces three different contingencies sets and in particular the following events combinations

$$\Theta_{J_1} = \{H_j, L_j\}$$

$$\Theta_{J_2} = \{H_j H_{j'}, H_j L_{j'}, L_j H_{j'}, L_j L_{j'}\}$$

$$\Theta_{J_3} = \{H_{j'} H_{j''}, H_{j'} L_{j''}, L_{j'} H_{j''}, L_{j'} L_{j''}\}$$

The implication mapping can be summarized as following

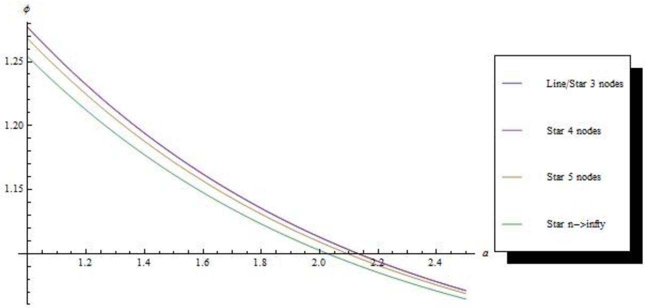
$w_1 \to \{H_j\}$

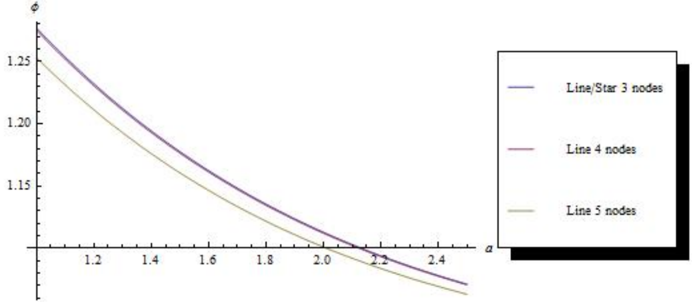
$w_{2,3} \to \{H_j, L_j\}$

$w_4 \to \{L_j\}$

$w_{5,8} \to \{H_j H_{j'}, L_j H_{j'}\}$

$w_{6,7} \to \{H_j H_{j'}, H_j L_{j'}, L_j H_{j'}, L_j L_{j'}\}$


ϕ
1.25
1.20
1.15
1.2
1.4
1.6
1.8
2.0
2.2
2.4
a
Line/Star 3 nodes
Star 4 nodes
Star 5 nodes
Star n–>infty


ϕ
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1.15
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1.6
1.8
2.0
2.2
2.4
a
Line/Star 3 nodes
Line 4 nodes
Line 5 nodes

Lemma

Given the assumptions above and an ambiguity averse DM, increasing the average distance between a node i and the rest of the nodes belonging to the same component increases the ambiguity perceived by i.

Lemma

The player of the component with the highest closeness centrality score is the one facing the lowest ambiguity among the players of the same component.

Static Analysis

We can use two different measures to study the ambiguity level of a given network structure g:

$$Cl_g = 1 - \frac{1}{\tilde{d}_g} \in [0, 1)$$

$$D_g = \frac{2(L_g - N + 1)}{N^2 - 3N + 2} \in [0, 1]$$

Proposition

The connected network component of $N \geq 3$ nodes with the lowest ambiguity score is the Star-network.

Network Dynamics

A network $g' \in G$ is obtainable from $g \in G$ through deviations by $S \subset N$ if

- $l_{ij} \in g'$ and $l_{ij} \notin g$ implies $i, j \subset S$, and
- $l_{ij} \in g$ and $l_{ij} \notin g'$ implies $i, j \cap S \neq \emptyset$

with S defining a coalition, subset of N nodes.

Each agent $i \in N$ optimally chooses $g^* \in \mathbb{G}_i$, the set of all the graphs obtainable by i modifying his starting link-structure, such that

$$y_i(g^*) > y_i(g'), \; \forall g' \neq g^*$$

where $y_i(g) \equiv \mathbb{E}_\mu \phi_i(\mathbb{E}_\pi u_i \circ f_g) - c_i(g)$.

Proposition

Assuming tree-graphs, a strongly stable network exists only for relatively high cost per link c. If it is the case it will be composed by $\frac{N}{2}$ dyads.

Proposition

Allowing for cycles in the network, if a strongly stable network exists, it will be composed of one or more bounded in size complete components.

Proposition

If a strongly stable network exists, it will also be necessarily efficient.

Conclusion

- The complexity of a network structure describing the influence-relationships between agents could creates extra *uncertainty*.
- The "degree" of uncertainty faced by each agent is related to the link-structure and in particular to his relative location in the network.
- The endogenous formation of "small" communities or groups of risk-sharing agents can be explained by a positive ambiguity aversion of the players involved.

Extensions

- Heterogeneity on ambiguity aversion coefficients, and directed links.
- Myopic players and "disclosure" of the real network g.

The End